Using mathematical induction, prove that $1(4)+2(7)+3(10)+\cdots+n(3n+1)=n(n+1)^2$ for all integers $n \geq 1$.

SCORE: _____ / 15 PTS

You must **NOT** use the shortcut formulae for the sums of the positive integers nor their squares.

(1) (1) (1)

BASIS STEP: $1(4) = 4 = 1(2)^2$ ✓

INDUCTIVE STEP: ASSUME $1(4)+2(7)+\cdots+k(3k+1) = k(k+1)^2$ (1)

FOR SOME PARTICULAR BUT ARBITRARY INTEGER $k \geq 1$ (1)

$1(4)+2(7)+\cdots+(k+1)(3(k+1)+1)$ (1)

$= 1(4)+2(7)+\cdots+k(3k+1)+(k+1)(3k+4)$ (1)

$= k(k+1)^2+(k+1)(3k+4)$ (1)

$= (k+1)[k(k+1)+3k+4]$

$= (k+1)(k^2+k+3k+4)$ (4)

$= (k+1)(k^2+4k+4)$ (1)

$= (k+1)(k+2)^2$ (1)

SO, $1(4)+2(7)+\cdots+n(3n+1) = n(n+1)^2$

FOR ALL INTEGERS $n \geq 1$ (1)

Consider the expansion of $(7x^4 + 5y^3)^{19}$. **SCORE: _____ / 8 PTS**

For the questions below, you may write the coefficients in your final answers in factored form, as shown in lecture.

The coefficients in your final answers must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

[a] Find the first 3 terms in the expansion. Simplify all signs, exponents and coefficients as shown in lecture.

$$\binom{19}{0}(7x^4)^{19}(5y^3)^0 + \binom{19}{1}(7x^4)^{18}(5y^3)^1 + \binom{19}{2}(7x^4)^{17}(5y^3)^2$$

$$= 7^{19}x^{76} + 19\cdot 7^{18}\cdot 5x^{72}y^3 + 19\cdot 9\cdot 7^{17}\cdot 5^2 x^{68}y^6$$

$$\binom{19}{2} = \frac{19!}{2!\,17!} = \frac{19\cdot \cancel{18}^{9}\cdot \cancel{17!}}{\cancel{2}\cdot 1\cdot \cancel{17!}} = 19\cdot 9$$

ALL ITEMS $\frac{1}{2}$ POINT EACH

[b] Find the 16^{th} term in the expansion. Simplify all signs, exponents and coefficients as shown in lecture.

$$\binom{19}{15}(7x^4)^{19-15}(5y^3)^{15}$$

$$= 19\cdot 6\cdot 17\cdot 2\cdot(7x^4)^4\cdot(5y^3)^{15}$$

$$= 19\cdot 6\cdot 17\cdot 2\cdot 7^4\cdot 5^{15}x^{16}y^{45}$$

$$\binom{19}{15} = \frac{19!}{15!\,4!} = \frac{19\cdot \cancel{18}^{6}\cdot 17\cdot \cancel{16}^{2}\cdot \cancel{15!}}{\cancel{15!}\,\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1} = 19\cdot 6\cdot 17\cdot 2$$

Expand $(3\sqrt{t}-\frac{2}{t^2})^5$ and simplify all signs, exponents and binomial coefficients. **SCORE: _____ / 7 PTS**

You may write the coefficients in your final answer in factored form, as shown in lecture.
The coefficients in your final answers must **NOT** contain division, ! nor $C(n,r)$ (nor equivalent) notation.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

$$
\begin{aligned}
&= 1\cdot(3t^{\frac{1}{2}})^5(-2t^{-2})^0 && = \underline{3^5 t^{\frac{5}{2}}} \; \tfrac{1}{2} && = 243t^{\frac{5}{2}} \\
&+ 5\cdot(3t^{\frac{1}{2}})^4(-2t^{-2})^1 && \; 1 \; \underline{-5\cdot 3^4\cdot 2}\,t^2 t^{-2} && -\underline{810} \; \tfrac{1}{2} \\
&+ 10\cdot(3t^{\frac{1}{2}})^3(-2t^{-2})^2 && \; 1 \; \underline{+10\cdot 3^3\cdot 2^2}\,t^{\frac{3}{2}} t^{-4} && +1080\underline{t^{-\frac{5}{2}}} \; \tfrac{1}{2} \\
&+ 10\cdot(3t^{\frac{1}{2}})^2(-2t^{-2})^3 && \; 1 \; \underline{-10\cdot 3^2\cdot 2^3}\,t\,t^{-6} && -720\underline{t^{-5}} \; \tfrac{1}{2} \\
&+ 5\cdot(3t^{\frac{1}{2}})^1(-2t^{-2})^4 && \; 1 \; \underline{+5\cdot 3\cdot 2^4}\,t^{\frac{1}{2}} t^{-8} && +240\underline{t^{-\frac{15}{2}}} \; \tfrac{1}{2} \\
&+ 1\cdot(3t^{\frac{1}{2}})^0(-2t^{-2})^5 && \; \tfrac{1}{2} \; \underline{-2^5 t^{-10}} && -32t^{-10}
\end{aligned}
$$